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A theory of turbulence in stratified fluids

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An effort is made to understand turbulence in fluid systems like the oceans and atmosphere in which the Richardson number is generally large. Toward this end, a theory is developed for turbulent flow over a flat plate which is moved and cooled in such a way as to produce constant vertical fluxes of momentum and heat. The theory indicates that in a co-ordinate system fixed in the plate the mean velocity increases linearly with height z above a turbulent boundary layer and the mean density decreases as z^3, so that the Richardson number is large far from the plate. Near the plate, the results reduce to those of Monin & Obukhov.

The *curvature* of the density profile is essential in the formulation of the theory. When the curvature is negative, a volume of fluid, thoroughly mixed by turbulence, will tend to flatten out at a new level well above the original centre of mass, thereby transporting heat downward. When the curvature is positive a mixed volume of fluid will tend to fall a similar distance, again transporting heat downward. A well-mixed volume of fluid will also tend to rise when the density profile is linear, but this rise is negligible on the basis of the Boussinesq approximation. The interchange of fluid of different, mean horizontal speeds in the formation of the turbulent patch transfers momentum. As the mixing in the patch destroys the mean velocity shear locally, kinetic energy is transferred from mean motion to disturbed motion. The turbulence can arise in spite of the high Richardson number because the precise variations of mean density and mean velocity mentioned above permit wave energy to propagate from the turbulent boundary layer to the whole region above the plate. At the levels of reflexion, where the amplitudes become large, wave-breaking and turbulence will tend to develop.

The relationship between the curvature of the density profile and the transfer of heat suggests that the density gradient near the level of a point of inflexion of the density curve (in general cases of stratified, shearing flow) will increase locally as time goes on. There will also be a tendency to increase the shear through the action of local wave stresses. If this results in a progressive reduction in Richardson number, an ultimate outbreak of Kelvin–Helmholtz instability will occur. The resulting sporadic turbulence will transfer heat (and momentum) through the level of the inflexion point. This mechanism for the appearance of regions of low Richardson number is offered as a possible explanation for the formation of the surfaces of strong density and velocity differences observed in the oceans and atmosphere, and for the turbulence that appears on these surfaces.

1. Introduction

A problem of great interest in studies of the atmosphere and oceans is the relationship between turbulence and the gravitational stability of these fluid systems. If we restrict consideration to smaller disturbances in which vertical velocities are an appreciable fraction of horizontal velocities, a relevant measure of stability is the Richardson number,

$$Ri = -\frac{g}{\rho_0}\frac{\partial \bar{\rho}_1}{\partial z}\bigg/\left(\frac{\partial \bar{u}}{\partial z}\right)^2,$$

where z is a vertical distance, ρ_0 is a representative density, $\bar{\rho}_1$ is the mean density, and $\bar{u}$ is the mean horizontal velocity. We understand that ρ refers to potential density when the fluid is compressible. In addition to Ri, expressed in terms of the mean gradients at a level z, we also refer to an *overall* Richardson number, Ri^* calculated for a whole layer of fluid in terms of the density and velocity gradients averaged over the layer. The symbol Ri' represents a *local* Richardson number, measured in terms of the local density gradient and local shear in the disturbed motion.

The orders of magnitude of Ri in the oceans and atmosphere tend to be large and, according to theories of instability in stratified shearing currents dating back to Taylor (1931*a*), a large Richardson number is strongly stabilizing with respect to the growth of small disturbances. Yet turbulence is a characteristic feature of the oceans and atmosphere. An example is the analysis by Taylor (1931*b*) of observations in the Kattegat by Jacobsen showing a diffusion of salt thousands of times greater than that corresponding to molecular processes at Richardson numbers of 100 or more. Also, smoke-puff experiments by Kellogg (1956) led Stewart (1959) to infer that the atmosphere is at least weakly turbulent everywhere except possibly in very local regions. More recent observations (Woods 1968; Reiter 1969) show that strong or moderate turbulence in regions outside the turbulent boundary layers is of a patchy nature and occurs near thin layers or sheets of strong vertical shear and density gradient in which Ri is of order one. Apparently, some mechanism exists that creates layers of strong velocity and density gradients in which Kelvin–Helmholtz instability develops.

The nature of the turbulence in the regions between the sheets observed by Woods and the layers of clear-air turbulence observed by aircraft is the subject of controversy. Such regions may be laminar everywhere or, as indicated by Kellogg's observations, weakly turbulent everywhere. On the other hand, Woods (private communication) suggests on the basis of preliminary observations in the oceans that such regions are generally laminar but with occasional patches of turbulence. If, as seems likely, these regions have Richardson numbers larger than $\frac{1}{4}$, the origin of the turbulence that may exist is different from the growing and breaking of small waves as occurs in Kelvin–Helmholtz instability (Miles 1963; Miles & Howard 1964). Indeed wave-breaking and turbulence can occur at large Richardson numbers if there is an appreciable source of disturbances. For example, the passing of air over mountains and hills can cause the formation and breaking of internal waves in the troposphere and stratosphere. These breaking waves occur also in laboratory experiments (Long 1955) in which the overall Richardson number is effectively infinite. It seems very likely that many sources of disturbances can cause waves to form and break in regions of large Richardson number; one example is the motion and acceleration of frontal surfaces in the atmosphere. Another may be the disturbances in the turbulent boundary layers of the oceans and atmosphere.

The present paper contains a theoretical analysis of an idealized experiment in which turbulence of a patchy nature is presumed to exist in regions of large Richardson numbers. As mentioned above, the turbulence may come from breaking waves which originate from disturbances in the turbulent boundary layer.

2. Idealized experimental model

Accordingly, consider the problem of an infinite, turbulent liquid in mean motion along the x axis over a cooled, smooth plate at $z = 0$. Originally, there may be two plates a distance H apart, moving in opposite directions along the x axis, heated above and cooled below to produce a stable vertical density distribution and a weak vertical shear. If the motion is statistically steady, and if mean quantities do not vary horizontally, the vertical fluxes of heat and momentum will be constant. If, now, the upper plate is moved further and further away, but the speeds and temperatures of the plates are adjusted to yield the same fluxes, we may, conceivably, neglect the upper plate for $z \ll H$.

With the Boussinesq approximation, the equations of motion, continuity, and heat conduction are

$$\frac{d\mathbf{v}}{dt} = -\nabla p - \rho\mathbf{k} + \nu\nabla^2\mathbf{v}, \tag{1}$$

$$\nabla.\mathbf{v} = 0, \tag{2}$$

$$\frac{d\rho}{dt} = K\nabla^2\rho, \tag{3}$$

where $\mathbf{v}\,(u, v, w)$ is the vector velocity, ν is the viscosity, K is the conductivity, and ρ is related to density ρ_1 and some representative density ρ_0 by the equation,

$$\rho_1 = \rho_0 + \frac{\rho_0}{g}\rho. \tag{4}$$

For the sake of simplicity, we will consider K and ν to be of the same order.

When there are two plates at $z = 0$ and $z = H$, the parameters of the problem are ν, K, H, Δu, and $\Delta\rho$, where $-\Delta u$ and Δu are the speeds of the lower and upper plates along the x axis, and where the plates are maintained at 'densities' $\Delta\rho$ and $-\Delta\rho$, respectively. The quantities,

$$\tau = \nu\bar{u}_z - \overline{u'w'} \tag{5}$$

and

$$q = K\bar{\rho}_z - \overline{w'\rho'}, \tag{6}$$

are proportional to the vertical fluxes of horizontal momentum and heat, and are constant in space and time. One of our basic assumptions is that if ν and K, are arbitrarily small, the flow is turbulent at almost all levels in the channel, even in cases of large Ri^*, in the sense that the fluxes of momentum and heat are dominated almost everywhere by the second terms on the right of (5) and (6). The assumption of turbulence appears to be unreasonable at first glance because of the strong gravitational stability associated with large Richardson numbers. We show later that the fluid can assume certain distributions of mean density and mean velocity that permit disturbances from the turbulent boundary layers to grow and develop into turbulence in regions of large *Ri*.

Turbulent fluxes

In the experiment with the two heated, moving plates, the overall Richardson number is

$$Ri^* = \frac{H\Delta\rho}{(\Delta u)^2}. \tag{7}$$

As already discussed, we will assume that this number is large. We now define the length $l \equiv (\Delta u)^2/\Delta\rho$. Since l/H is small, the most reasonable assumption for τ and q is a power-law dependence on this number, i.e.

$$\tau \sim (\Delta u)^2 \left(\frac{l}{H}\right)^{s_1}, \quad q \sim \Delta u \Delta\rho \left(\frac{l}{H}\right)^{s_2}, \tag{8}$$

where s_1 and s_2 are unknown exponents. It is reasonable to assume that u' and w' do not exceed Δu and that ρ' does not exceed $\Delta\rho$ in order of magnitude; then s_1 and s_2 are non-negative. Indeed, we anticipate generally weak and sporadic turbulence in the interior at large *Ri*, so that correlations between w', u' and between w', ρ' are expected to be low. We assume, therefore, that s_1 and s_2 are both positive. Let us now increase H and adjust Δu and $\Delta\rho$ accordingly to maintain the same τ and q. As H gets very large, we suppose that H will no longer affect conditions at heights $z \ll H$, so that we may adopt τ, q, z, ν and K as the only quantities that determine conditions near the plate. In physical terms, this assumption implies that disturbances have dimensions much less than H when the Richardson number is large.

3. Viscous sublayer at a smooth plate

In §3 and in §4, we will analyze the boundary layers over a flat plate at $z = 0$. We will assume that the plate is smooth so that molecular effects will be important sufficiently close to $z = 0$. Near the plate, therefore, τ and q are given by the molecular terms in (5) and (6). From boundary-layer theory, the thickness of the viscous layer l_b is such that $u_b l_b \sim \nu$, where u_b is the mean speed just outside. But since τ is of the order of the first term on the right of (5), $u_b \sim \tau l_b/\nu$, so that $l_b \sim \nu/\tau^{\frac{1}{2}}$. Therefore, assuming $\nu \sim K$, we have, as a first approximation in the viscous layer,

$$\bar{u}_z = \frac{\tau^{\frac{1}{2}}}{l_b} f_1\left(\frac{z}{l_b}\right), \quad \bar{\rho}_z = \frac{q}{\tau^{\frac{1}{2}} l_b} g_1\left(\frac{z}{l_b}\right), \tag{9}$$

where f_1, g_1 and z/l_b are of order one.†

The Richardson number is proportional to ν and is very small in this layer. As a result, the density variation has a negligible dynamical effect, and the source of vorticity for disturbances that may exist is the basic shear. Thus‡

$$u' \sim \bar{u}_z l_b \sim \tau^{\frac{1}{2}}.$$

†, ‡ For footnotes see facing page.

Since l_b is the only length unit in the viscous sublayer, the equation of continuity yields

$$u' \sim v' \sim w' \sim \tau^{\frac{1}{2}}, \tag{10}$$

$$l_1 \sim l_2 \sim l_3 \sim l_b, \tag{11}$$

where l_1, l_2, l_3 are the length scales of the disturbances in the directions of x, y, z. Finally, since density tends to be a convected quantity, we assume that $\rho' \sim \bar{\rho}_z l_b$, or

$$\rho' \sim \frac{q}{\tau^{\frac{1}{2}}}. \tag{12}$$

The above estimates make all terms in the governing equations of the same order, except the ρ' term in the vertical perturbation equation of motion. Its ratio to the first-order terms is l_b/l_m, where $l_m \equiv \tau^{\frac{3}{2}}/q$ is one of the fundamental lengths in the regions outside the viscous sublayer.

4. Turbulent boundary layer

At the outer edge of the laminar layer, the motion becomes turbulent, by supposition. We neglect molecular effects in this region. We also neglect the distance between the plates H compared to the length l_m. It follows that

$$\bar{u}_z = \frac{q}{\tau} f_2\left(\frac{z}{l_m}\right), \quad \bar{\rho}_z = \frac{q^2}{\tau^2} g_2\left(\frac{z}{l_m}\right) \tag{13}$$

in this region, where f_2, g_2, z/l_m, and the Richardson number are all of order one.

The turbulent scales also follow directly from dimensional analysis. Thus

$$u' \sim v' \sim w' \sim \tau^{\frac{1}{2}}, \quad \rho' \sim \frac{q}{\tau^{\frac{1}{2}}}, \quad l_1 \sim l_2 \sim l_3 \sim l_m. \tag{14}$$

If there is a region such that $l_b \ll z \ll l_m$, the assumption (Millikan 1938) that the solutions in regions $z \sim l_b$, $z \sim l_m$, have overlapping validity yields

$$\bar{u}_z = A_1 \frac{\tau^{\frac{1}{2}}}{z}, \quad \bar{\rho}_z = B_1 \frac{q}{\tau^{\frac{1}{2}} z}, \quad Ri \sim \frac{z}{l_m},$$

$$u' \sim v' \sim w' \sim \tau^{\frac{1}{2}}, \quad \rho' \sim \frac{q}{\tau^{\frac{1}{2}}}, \quad l_1 \sim l_2 \sim l_3 \sim z, \tag{15}$$

† The use of the symbolism $a \sim b$ or $a/b \sim 1$ is the same as in boundary-layer theory (see e.g. Schlichting 1955). More precisely, suppose we imagine that we have solved the problem for, say $\bar{u}_z$. We may write, quite generally,

$$\phi\left(\frac{\bar{u}_z}{q/\tau}, \frac{z}{l_m}, \epsilon_1, \epsilon_2\right) = 0,$$

where $l_m = \tau^{\frac{3}{2}}/q$, and $\epsilon_1 = l_b/l_m$, $\epsilon_2 = l_m/H$. If, in a certain range of z and for arbitrarily small ϵ_1, ϵ_2, this reduces to the form,

$$f\left(\frac{\bar{u}_z}{S_1 q/\tau}, \frac{z}{l_m S_2}\right) = 0,$$

where S_1, S_2 are functions of ϵ_1, ϵ_2, then we say that $\bar{u}_z \tau / S_1 q$, $z/l_m S_2$, are of order one. The same applies to other mean quantities.

‡ When we use a perturbation quantity in connexion with order-of-magnitude arguments, we really mean some appropriate averaged quantity, e.g. the root-mean-square of u' instead of u' itself.

where A_1 and B_1 are constants of order one. This is the logarithmic layer which also occurs in turbulent flow of a homogeneous fluid over a smooth or rough surface (Goldstein 1938).

In the logarithmic layer, the only term of lower order in the equations is again ρ', and its ratio to the first-order terms is of order

$$Ri \sim \frac{z}{l_m}. \tag{16}$$

This suggests that the full form of the solution for $\overline{u}_z$ in the logarithmic layer and above is

$$\overline{u}_z = \frac{\tau^{\frac{1}{2}}}{z}\left(A_1 + A_2\frac{z}{l_m} + \ldots\right).$$

This agrees with a suggestion of Monin & Obukhov (1954). We call l_m the *Monin–Obukhov length.*

5. The interior region

Let us now consider the interior region $z \sim H$. Here the length H is of fundamental importance, although the properties of the disturbance and mean fields must be those of the last section as we approach the plates.

A basic physical assumption for the boundary layers and now for the interior is that the length scales of the disturbances be much smaller than the height of the channel. This is related to the presence of density variations, of course, because, if $\Delta\rho = 0$, the only length that can determine the size of the disturbances is H itself. A non-zero $\Delta\rho$, on the other hand, introduces another length l into the analysis. In physical terms, the stability of the density field will tend to limit the vertical scale† of the disturbances.

The stability should also inhibit the outbreak of turbulence, and, with this in mind, we will assume, as a fundamental physical picture of conditions in the interior, that much of the region is in laminar motion in which energy dissipation and heat transfer are negligible. Here and there are turbulent patches where dissipation and heat flux occur.

According to classical stability arguments, small disturbances are stable when the Richardson number is large. It is quite apparent however, that finite disturbances can create conditions in which the local value of the Richardson number is less than the value computed from the mean fields, as large vorticities (large values of u'_z) are generated by the pressure-density effect. If the local Richardson number falls below $\frac{1}{4}$, small disturbances, superimposed on the finite disturbances, are likely to grow and lead to turbulence (Phillips 1966). Since we have assumed that turbulence exists in the interior, we will now assume that the amplitudes of the wave motion are large enough to yield local Richardson numbers of order one or less. Since $\overline{u}_z$ is small, the local Richardson number has the form.

$$\frac{\overline{\rho}_z + \rho'_z}{u'^2_z}.$$

† As we will see, it also reduces the horizontal scale of the disturbances.

We may estimate
$$\rho'_z \sim \frac{\bar{\rho}_z \delta}{l_3},$$

where δ is a length of the order of the wave amplitude, and l_3^{-1} is the order of the vertical derivative in the disturbed motion. Also, from the vorticity equation for the disturbed vorticity ζ', we get

$$\frac{d\zeta'}{dt} \sim \rho'_x, \tag{17}$$

so that
$$u'_z \sim \frac{N\delta}{l_1}, \tag{18}$$

where N is the Brunt–Väisälä frequency, i.e.

$$N = |\bar{\rho}_z|^{\frac{1}{2}}. \tag{19}$$

Thus the local Richardson number has the form,

$$\frac{1+A(\delta/l_3)}{\delta^2/l_1^2}, \tag{20}$$

where A is of order one.

We now estimate the disturbed velocity. The third equation of motion yields

$$w' \sim \rho' N^{-1} \sim N\delta, \tag{21}$$

and (18) yields
$$u' \sim N\frac{\delta l_3}{l_1}. \tag{22}$$

The equation of continuity now reveals that $l_3 \sim l_1$. The local Richardson number is of order

$$\frac{l_1^2}{\delta^2} + A\frac{l_1}{\delta}. \tag{23}$$

Thus, as the amplitude increases, the local Richardson number drops to order one when the amplitude becomes of the order of the wavelength. This indicates that turbulence in the interior will occur† if some of the waves become large, despite high values of Ri.

Curvature of the density profile

One of the fundamental concepts of this paper is the idea that finite amplitude waves tend to break down into turbulence, and that the resulting turbulent patches effect the heat transport from higher to lower levels in a way that depends utterly on the *curvature of the mean density profile*. We may approach this in two ways. The most direct is to suppose that a turbulent region of linear dimensions l_p is produced by breaking of internal gravity waves. We assume the patch is initially undisturbed with density given by $\bar{\rho}(z)$. It is then disturbed by waves which break and mix the patch thoroughly When this happens, it will tend to flatten out and seek the level at which the environmental density equals the

† Bretherton (1969) has suggested that the local Richardson number will drop to values of order one (less than $\frac{1}{4}$) in certain regions simply as the result of chance superpositions of waves.

average density of the patch. If the new level is different from the original level of the centre of mass of the material, heat will be transported vertically by this process. If z_m is taken at the centre of mass, the average value of ρ for the mixed patch is

$$\bar{\rho}_p = \bar{\rho}(z_m)+\bar{\rho}_z L+\bar{\rho}_{zz}\tfrac{1}{2}L^2+\ldots+A\bar{\rho}_{zz}l_p^2+\ldots \tag{24}$$

approximately, where A is of order one, and L is the height of the centroid above the centre of mass. The patch flattens out at a height L_p above or below the centre of mass given by

$$L_p \sim \frac{A\bar{\rho}_{zz}l_p^2}{\bar{\rho}_z}+L+\ldots. \tag{25}$$

The distance L is easily computed. Using (4), and taking $z = 0$ at the centroid, it is

$$L = \frac{\rho_0}{gM}\iiint \rho z\, dv,$$

where dv is the element of volume of the patch and M is its mass. Expanding $\rho = \bar{\rho}(z)$ about the centroid we get

$$L = \frac{\iiint (\bar{\rho}+\bar{\rho}_z z+\bar{\rho}_{zz}\frac{1}{2}z^2+\ldots)\, z\, dv}{g\iiint \left(1+\frac{\bar{\rho}}{g}+\frac{\bar{\rho}_z}{g}z+\frac{\bar{\rho}_{zz}}{g}\frac{z^2}{2}+\ldots\right) dv},$$

where $\bar{\rho}, \bar{\rho}_z, \bar{\rho}_{zz}$ are evaluated at the centroid. Evaluating, we get

$$L = \frac{Al_p^5\bar{\rho}_z+B\bar{\rho}_{zz}l_p^6+\ldots}{gCl_p^3\{1+(\bar{\rho}/g)\}+D\bar{\rho}_{zz}l_p^5+\ldots},$$

where A, B, C, D, ... are of order one. Since l_p is much less than the vertical scale of the mean motion, the dominant terms give

$$L \sim \frac{|\bar{\rho}_z|\, l_p^2}{g}. \tag{26}$$

The ratio of L to the first term in (25) is of order $\bar{\rho}/g$ or, using (4) of order $(\bar{\rho}_1-\rho_0)/\rho_0$. This is small of the order of terms neglected in the Boussinesq approximation, so that the vertical displacement of the patch, the mixing length of the process, is

$$L_p \sim \frac{\bar{\rho}_{zz}l_p^2}{\bar{\rho}_z}, \tag{27}$$

and is proportional to the curvature of the density profile. Notice that the patch rises when the curvature is negative and falls when the curvature is positive. In either case heat is transported downward.

The flux of heat may be computed by writing

$$q \sim A_K\bar{\rho}_z, \tag{28}$$

where A_K is the coefficient of eddy diffusion and is given by the product of the mixing length L_p, the vertical velocity of the patch $w_p \sim NL_p$, and the ratio R of the volume of the turbulent patches to the whole volume. Thus,

$$q \sim R\frac{\bar{\rho}_{zz}^2 l_p^4}{N}. \tag{29}$$

There is another way to bring out the importance of the curvature of the density profile. If we form the kinetic energy equation from (1), we get

$$\frac{\partial}{\partial t}\left(\frac{c^2}{2}\right)+\nabla.\left[\mathbf{v}\left(p+\frac{c^2}{2}\right)\right] = -\rho w+\nu\mathbf{v}.\nabla^2\mathbf{v}, \tag{30}$$

where c is the fluid speed. If we expand (30) in terms of mean and perturbation quantities and average, we get

$$\frac{\partial}{\partial t}\overline{\left(\frac{c'^2}{2}\right)} = -\frac{\partial}{\partial z}\left[\overline{w'\left(\frac{c'^2}{2}+p'\right)}\right]-\overline{w'u'}\,\overline{u}_z-\overline{w'\rho'}-\epsilon, \tag{31}$$

where we have used
$$\overline{u}_t = \nu\overline{u}_{zz}-(\overline{u'w'})_z, \tag{32}$$

and where
$$\epsilon = \nu[\overline{(\nabla u')^2+(\nabla v')^2+(\nabla w')^2}]-\nu\frac{\partial^2}{\partial z^2}\overline{\left(\frac{c'^2}{2}\right)} \tag{33}$$

is dominated by the positive first term and is related to viscous dissipation. Equation (31) expresses the rate of increase of the average disturbance kinetic energy as a sum of four terms. The first is an advection of energy through the boundary of the region in question. It is easily seen that it is very small in the ratio l_m/H compared to other terms in the energy equation when integrated over the whole of the interior region. The last term on the right-hand side of (31) is substantially equal to the loss of energy by viscous dissipation in very small eddies.

The term $-\overline{w'u'}\,\overline{u}_z$ represents a gain of kinetic energy of the disturbance at the expense of the mean motion. Thus, if a portion of fluid is thoroughly mixed, its mean motion will tend to become uniform with height, and the wiping-out of the mean shear makes energy available for the disturbances. If we use the concept of turbulent patches, the mixing of the high-velocity fluid in the upper portion of the patch with the low-velocity fluid in the lower portion of the patch reduces the energy of the mean field by an amount of order $\overline{u}_z^2 l_p^2$.

The term $-\overline{w'\rho'}$ in (31) represents a loss of disturbance kinetic energy. The interpretation by Richardson (1920) and others is that it represents an increase of potential energy by turbulence as heavy fluid tends to rise and light fluid tends to fall to effect the heat transfer. Richardson and subsequent investigators assume that $\overline{w'u'}\,\overline{u}_z$ and $\overline{w'\rho'}$ are of the same order, since each represents a fundamental process in the turbulent motion of a stratified, shearing current.

The relationship of the $\overline{w'\rho'}$ term to potential energy is important to our discussion. Thus, consider a particle in a stratified medium at a height ξ above the level z_0 at which the mean density of the fluid equals the particle density, i.e. $\xi = z-z_0$, and $\rho = \overline{\rho}(z_0)$. Neglecting pressure effects, we have

$$\frac{d^2\xi}{dt^2} = -\rho' = -\rho+\overline{\rho}(z) = -\overline{\rho}(z_0)+\overline{\rho}(z) \simeq \overline{\rho}_z\xi, \tag{34}$$

whence, approximately,
$$\frac{1}{2}\left(\frac{d\xi}{dt}\right)^2-\overline{\rho}_z\frac{\xi^2}{2} = \text{constant}. \tag{35}$$

We call
$$V' = -\overline{\rho}_z\frac{\xi^2}{2} \simeq -\frac{\rho'^2}{2\overline{\rho}_z} \tag{36}$$

the *available potential energy* of the disturbance. Let us now multiply the disturbance heat equation by ρ' and average. We get

$$\frac{\partial}{\partial t}\left(\frac{\overline{\rho'^2}}{2}\right) = -\frac{\partial}{\partial z}\left(\frac{\overline{w'\rho'^2}}{2}\right) - \overline{w'\rho'}\,\bar{\rho}_z - \delta, \tag{37}$$

where the dissipation function,

$$\delta = -K\frac{\partial^2}{\partial z^2}\left(\frac{\overline{\rho'^2}}{2}\right) + K\overline{(\nabla\rho')^2}, \tag{38}$$

is dominated by the positive second term on the right. Assuming a steady state, (37) may be written

$$\frac{\partial(\overline{w'V'})}{\partial z} - \frac{\bar{\rho}_{zz}\overline{w'\rho'^2}}{2\bar{\rho}_z^2} - \overline{w'\rho'} = -\frac{\delta}{|\bar{\rho}_z|}. \tag{39}$$

If we now combine this with the steady-state energy equation (31), we get

$$\frac{\partial}{\partial z}\left[\overline{w'\left(p' + \frac{c'^2}{2} + V'\right)}\right] + \overline{w'u'}\,\bar{u}_z - \rho_{zz}\frac{\overline{w'\rho'^2}}{2\bar{\rho}_z^2} = -\epsilon - \frac{\delta}{|\bar{\rho}_z|}. \tag{40}$$

The $\bar{\rho}_{zz}$ term may be related to the concept of turbulent patches by assuming that contributions to $\overline{w'\rho'^2}$ come principally from the vertical motion of the patches, $w_p \sim L_p N$, so that

$$\overline{w'\rho'^2} \sim Rw_p\overline{\rho'^2}. \tag{41}$$

Since w_p changes sign with $\bar{\rho}_{zz}$, the curvature term in (40) always acts as a sink of energy as opposed to the source of energy in the shear term. If we evaluate the curvature term from the transfer in the patches, we get

$$\frac{\bar{\rho}_{zz} R L_p N \rho'^2}{\bar{\rho}_z^2} \sim \bar{\rho}_{zz} R \frac{\bar{\rho}_{zz} l_p^2 N \bar{\rho}_z^2 l_p^2}{\bar{\rho}_z^3} \sim R\frac{\bar{\rho}_{zz}^2 l_p^4}{N}. \tag{42}$$

Since this is the same as (29), we see that the $\bar{\rho}_{zz}$ term in (40) is of the order of q or, presumably, of the order of $\overline{u'w'}\,\bar{u}_z$. This again suggests the essential importance of curvature of the density profile in the turbulent transfer mechanism.

Calculation of orders of magnitude

Let us now estimate the order of magnitude of the various mean quantities and perturbation quantities. We assume the following for the interior of the channel:

$$\left.\begin{aligned}
&\bar{u}_z = \frac{q}{\tau}\left(\frac{H}{l_m}\right)^n f_3\left(\frac{z}{H}\right), \quad \bar{\rho}_z = \frac{q^2}{\tau^2}\left(\frac{H}{l_m}\right)^m g_3\left(\frac{z}{H}\right), \quad Ri \sim \left(\frac{H}{l_m}\right)^{m-2n},\\
&u' \sim v' \sim w' \sim \tau^{\frac{1}{2}}\left(\frac{H}{l_m}\right)^a, \quad l_1 \sim l_2 \sim l_3 \sim H\left(\frac{H}{l_m}\right)^{a-1-\frac{1}{2}m}, \quad \rho' \sim \frac{q^2}{\tau^2}H\left(\frac{H}{l_m}\right)^{\frac{1}{2}m+a-1},\\
&R \sim \left(\frac{H}{l_m}\right)^{-2-\frac{3}{2}m-4d}, \quad L_p \sim H\left(\frac{H}{l_m}\right)^{2d}, \quad l_p \sim H\left(\frac{H}{l_m}\right)^{d}, \quad w_p \sim \tau^{\frac{1}{2}}\left(\frac{H}{l_m}\right)^{2d+1+\frac{1}{2}m},\\
&u'_e \sim v'_e \sim w'_e \sim \tau^{\frac{1}{2}}\left(\frac{H}{l_m}\right)^{i}, \quad l_e \sim H\left(\frac{H}{l_m}\right)^{j}, \quad \rho'_e \sim \frac{q^2}{\tau^2}H\left(\frac{H}{l_m}\right)^{s}.
\end{aligned}\right\} \tag{43}$$

where the first set of perturbation quantities refers to the wave motion, and the second set, with subscript 'e', refers to the turbulent motion in the patches. Some of the exponents of H/l_m have been related by using the results we have already obtained, namely,

$$l_1 \sim l_3 \sim \delta, \quad w' \sim N\delta, \quad L_p \sim \frac{\bar{\rho}_{zz} l_p^2}{\bar{\rho}_z}, \quad w_p \sim L_p N, \quad q \sim R\frac{\bar{\rho}_{zz}^2 l_p^4}{N}, \quad \rho' \sim \bar{\rho}_z \delta, \tag{44}$$

by the assumption that the horizontal scales, l_1, l_2, are equal, by the equation of continuity, and by the assumption of equal length scales for the turbulent motion in the patches.

The remaining exponents may be found by using the following set of assumptions:

(i) We assume that the waves in the fluid of length l_3, originate either in the turbulent boundary layer, and therefore have a length l_m, or in the collapse of the patches. We also assume that the patches are formed by the breaking of individual waves of length l_3. Accordingly, we use

$$l_3 \sim l_m, \quad l_p \sim l_m. \tag{45}$$

(ii) When the fluid mixes in the patches, momentum is transferred from the upper portions to the lower portions, and this contributes to the momentum flux τ. Although, as we discuss later, momentum can be transferred by the wave motion itself, we assume that the turbulent transport is a significant portion of the total. Thus, we assume

$$\tau \sim A_\nu \bar{u}_z, \tag{46}$$

where the coefficient of eddy viscosity A_ν is the product of R, the mixing length for momentum l_p, and the vertical speed Nl_p, i.e.

$$A_\nu \sim RNl_p^2. \tag{47}$$

(iii) As mentioned earlier, we assume either†

$$\tau\bar{u}_z \sim q \tag{48}$$

in the kinetic energy equation (31), or

$$\tau\bar{u}_z \sim \bar{\rho}_{zz}\frac{\overline{w'\rho'^2}}{\bar{\rho}_z^2} \tag{49}$$

in the total energy equation (40).

Assumptions (i)–(iii) yield

$$n = 0, \quad d = -1, \quad m = 2, \quad a = 1. \tag{50}$$

(iv) Mixing in the turbulent patches destroys the mean shear locally. This yields a net energy increment for the disturbances of $\bar{u}_z^2 l_p^2$. This appears to be the basic process by which disturbance kinetic energy increases at the expense of the mean current. We assume that this goes directly into the kinetic energy of the eddy motion in the patches, i.e.

$$u_e'^2 \sim \bar{u}_z^2 l_p^2. \tag{51}$$

† G. I. Taylor, in a discussion of a paper by Stewart (1959), refers to the ratio $q/\tau\bar{u}_z$ as measured in the Kattegat. He found that the ratio approached one as the flow became more and more stable.

Of course, in the breaking process, the fluid particles fall a distance of order l_p, and the conversion of potential energy into kinetic energy yields speeds of order Nl_p. This however, is wave motion rather than turbulent motion.†

(v) We may picture the turbulent eddies of length l_e superimposed on the larger-scale wave motion of length l_3. The local density gradients in the wave motion are of order ρ'_z, and this means that the density fluctuations in the turbulent eddies are

$$\rho'_e \sim \rho' \frac{l_e}{l_3}. \tag{52}$$

(vi) We assume, according to observations in the atmosphere (Vinnichenko & Dutton 1969), that the spectra of the small-scale (turbulent) velocity and density fields follow the $k^{-\frac{5}{3}}$ law. Thus, let us assume the spectrum $E(k)$ for velocity fluctuations and the spectrum $F(k)$ for density fluctuations depend only on k, ϵ, and δ. If we demand a $k^{-\frac{5}{3}}$ behaviour, dimensional analysis yields

$$E(k_e) \sim \epsilon_p^{\frac{2}{3}} k_e^{-\frac{5}{3}}, \tag{53}$$

$$F(k_e) \sim \frac{\delta_p}{\epsilon_p^{\frac{1}{3}}} k_e^{-\frac{5}{3}}, \tag{54}$$

where ϵ_p and δ_p are values of the dissipation functions evaluated in the turbulent patches. Since

$$u_e'^2 \sim \int_{k_e}^{\infty} E(k)\, dk \sim \epsilon_p^{\frac{2}{3}} l_e^{\frac{2}{3}}, \tag{55}$$

$$\rho_e'^2 \sim \int_{k_e}^{\infty} F(k)\, dk \sim \frac{\delta_p}{\epsilon_p^{\frac{1}{3}}} l_e^{\frac{2}{3}}, \tag{56}$$

we find

$$\epsilon_p \sim \frac{u_e'^3}{l_e}, \tag{57}$$

$$\frac{\delta_p^3}{\epsilon_p} \sim \frac{\rho_e'^6}{l_e^2}, \tag{58}$$

or

$$\epsilon \sim R\frac{u_e'^3}{l_e}, \tag{59}$$

$$\delta \sim R\frac{u'_e \rho_e'^2}{l_e}. \tag{60}$$

If we integrate (37) between the two boundary layers, we get

$$\delta \sim \frac{q^3}{\tau^2}\left(\frac{H}{l_m}\right)^2, \tag{61}$$

so that (60) is

$$\frac{u'_e \rho_e'^2}{l_e} \sim \frac{q^3}{\tau^2}\left(\frac{H}{l_m}\right)^3. \tag{62}$$

Equations (51), (52) and (62) yield

$$i = 0, \quad s = 0, \quad j = -2. \tag{63}$$

Notice that this leads to $\epsilon \sim q$.

† A recent paper by Wu (1969) reports on an experimental study of the collapse of well-mixed patches of fluid in a stratified fluid. His observations support a collapse speed of order Nl_p in the initial and principal stages.

Our results summarize as follows:

$$\bar{u}_z \sim \frac{q}{\tau}, \quad \bar{\rho}_z \sim \frac{q^2}{\tau^2}\left(\frac{H}{l_m}\right)^2, \quad u' \sim v' \sim w' \sim \tau^{\frac{1}{2}}\frac{H}{l_m}, \quad Ri \sim \left(\frac{H}{l_m}\right)^2,$$

$$\left.\begin{aligned} l_1 \sim l_2 \sim l_3 \sim l_m, \quad \rho' \sim \frac{q}{\tau^{\frac{1}{2}}}\left(\frac{H}{l_m}\right)^2, \quad R \sim \frac{l_m}{H}, \\ L_p \sim \frac{l_m^2}{H}, \quad l_p \sim l_m, \quad w_p \sim \tau^{\frac{1}{2}}, \quad u_e' \sim v_c' \sim w_e' \sim \tau^{\frac{1}{2}}, \\ \rho_e' \sim \frac{q}{\tau^{\frac{1}{2}}}\frac{H}{l_m}, \quad l_e \sim \frac{l_m^2}{H}. \end{aligned}\right\} \quad (64)$$

If the upper boundary is absent so that the flow is over a single cooled plate at $z = 0$, the quantity H is missing from the analysis. All of the results in (64) still apply, however, if we replace H by z. For example,

$$\bar{u}_z = A_1\frac{q}{\tau}, \quad \bar{\rho}_z = A_2\frac{q^4}{\tau^5}z^2, \quad (65)$$

where A_1, A_2 are constants of order one†. The expressions in (64) with H replaced by z also hold in the two-plate problem in regions whose distance from the plate is much greater than l_m and much less than H.

Wave disturbances in the interior

The result that the mean velocity gradient is constant and that the mean density gradient varies as z^2 in a region above the turbulent boundary layer may be related to the problem of breaking waves in the interior of the fluid at high Richardson number. We have shown that turbulence can occur despite the stability if there are finite-amplitude waves, but the source of the wave energy was not discussed in detail. There are two sources, one from disturbances in the turbulent boundary layer and one from the collapse of turbulent patches. Thus, consider a wave-packet originating near $z = 0$ from eddy activity in the turbulent boundary layer. As shown by Booker & Bretherton (1967) and by Townsend (1968) some of the wave packets will propagate upwards to reflexion levels z_r before being reflected downward. For a wave of frequency σ relative to the plate (frequency and horizontal wavelength remain unchanged when packets move through a fluid of high Richardson number) this corresponds to a height at which the local Brunt–Väisälä frequency $N(z_r)$ equals the frequency relative to the fluid, $\sigma + \bar{u}(z_r)/l'$, where l' is the wavelength. Since the frequency of boundary-layer disturbances is of order q/τ, we have

$$A\frac{q}{\tau} + \frac{\bar{u}(z_r)}{l'} = N(z_r),$$

where $A \sim 1$. If we use the results in (65) we get

$$z_r \sim \frac{l''}{(l''/l_m) - 1}, \quad (66)$$

† Monin (1969) has found verification of the prediction in (65) for the density field. His analysis of 40 hydrological stations in the central North Pacific shows that the law $N(z) = Az$ is fulfilled quite satisfactorily in layers of the deep ocean below a depth of 1·5–2 km.

where l'' is of the order of the wavelength. Since $l'' \sim l_m$ for boundary-layer disturbances, wave energy in significant amounts can propagate to all levels. Similarly, we can show that wave energy generated by collapsing eddies can also propagate to all levels.

A more detailed investigation may be based on a solution of the problem of small disturbances in a stratified flow when the basic velocity and density distributions are those of the turbulent model. Thus, we assume

$$\bar{u} = \beta z,$$

$$\bar{\rho}_z = -\alpha^2 z^2.$$

The linearized differential equation for the vertical velocity is (Phillips 1966, p. 179)

$$D^2\nabla^2 w + \alpha^2 z^2 \frac{\partial^2 w}{\partial x^2} = 0, \tag{67}$$

where the operator is $$D = \frac{\partial}{\partial t} + \beta z \frac{\partial}{\partial x}.$$

For the disturbances of the form,

$$w = W(z)\, e^{i(kx - \sigma t)},$$

(67) becomes $$W'' + \left[\frac{\alpha^2 z^2 k^2}{(\beta z k - \sigma)^2} - k^2\right] W = 0. \tag{68}$$

An approximate solution of (68) may be obtained through the WKB approximation. It is

$$W = A \frac{(\beta z k - \sigma)^{\frac{1}{2}}}{[\alpha^2 z^2 k^2 - (\beta z k - \sigma)^2 k^2]^{\frac{1}{4}}} \exp i \left\{ \int_0^z \left[\frac{\alpha^2 z^2 k^2}{(\beta z k - \sigma)^2} - k^2\right]^{\frac{1}{2}} dz \right\}, \tag{69}$$

where A is a constant. The reflexion level of the wave corresponds to the level at which the vertical wave-number becomes zero or, where

$$\alpha^2 z_r^2 = (\beta z_r k - \sigma)^2.$$

Taking frequencies appropriate to waves originating in the turbulent boundary layer, i.e. $\sigma \sim q/\tau$, and values of α and β corresponding to (65), we again obtain (66) for the reflecting waves. Although the approximation fails at the reflexion level, (69) indicates that the wave amplitudes will be large near the reflexion level, suggesting a tendency for breakdown into turbulence. This may be the process by which the turbulent patches are formed.

Relationship of $\Delta u, \Delta \rho$ to τ, q

It is of interest to relate the parameters τ, q to the external parameters Δu, $\Delta \rho$ in the problem of two plates. It is apparent that the velocity and density increments over the laminar and turbulent boundary layers are small compared to those in the interior. In the interior, we have

$$\bar{u}_z \sim \frac{q}{\tau}, \quad \bar{\rho}_z \sim \frac{q^2}{\tau^2}\left(\frac{H}{l_m}\right)^2,$$

so that

$$\Delta u \sim \frac{q}{\tau}H, \quad \Delta\rho \sim H\frac{q^2}{\tau^2}\left(\frac{H}{l_m}\right)^2.$$

Therefore,

$$\frac{l}{H} \sim \left(\frac{l_m}{H}\right)^2, \tag{70}$$

and (8) becomes

$$\tau \sim (\Delta u)^2\left(\frac{l}{H}\right), \quad q \sim \Delta u \Delta\rho\left(\frac{l}{H}\right)^2. \tag{71}$$

In particular, notice that strong stability greatly decreases the drag at the two plates.

We may now compute the correlation coefficients c_1 and c_2 involved in τ and q. Using (64), we get

$$\tau \sim \overline{u'w'} \sim c_1 u'w' \sim c_1\tau\left(\frac{H}{l_m}\right)^2, \tag{72}$$

and

$$q \sim \overline{w'\rho'} \sim c_2 w'\rho' \sim c_2 q\left(\frac{H}{l_m}\right)^3. \tag{73}$$

Thus

$$c_1 \sim \left(\frac{l_m}{H}\right)^2 \sim \frac{l}{H}, \tag{74}$$

$$c_2 \sim \left(\frac{l_m}{H}\right)^3 \sim \left(\frac{l}{H}\right)^{\frac{3}{2}}. \tag{75}$$

The low value of the correlation coefficient c_2 reflects the fact that most of the region is in wave motion, which is incapable of transferring heat.

6. The formation of layers of low Richardson number

As discussed in § 1, there is some mechanism in the oceans and atmosphere that causes the formation of thin layers with large density and velocity gradients, but with low Richardson numbers. We may relate this to the theory of this paper in the following way. Suppose the turbulent flow between two plates is developing symmetrically from some initial state, and we postulate a density field in which the curvature is negative in the lower half of the channel and positive in the upper half of the channel (as demanded by the theory and by symmetry). There will be a rising and sinking of turbulent patches below and above the middle of the channel, but near the middle, the curvature will be small and the heat transport low, with a resulting tendency to form a strong gradient of density in this layer. If no other effect intervened, this would tend to make the middle layer even more stable and further inhibit turbulence and heat transport. A mechanism exists, however, for a buildup of the velocity gradient near the middle. Thus, when the turbulent patches collapse at the level z, waves are generated with wavelength $l' \sim l_m$ and frequency of order

$$\frac{\bar{u}(z)}{l'} \pm N(z) \tag{76}$$

relative to the bottom boundary. A portion of this wave energy associated with the upper sign moves upward to 'absorption' levels z_a, defined as the level at which the wave frequency is zero as measured by an observer moving at speed $\bar{u}(z_a)$ (Townsend 1968), i.e.

$$\frac{\bar{u}(z_a)}{l'} = \frac{\bar{u}(z)}{l'} + N(z). \tag{77}$$

The wave energy is lost to the mean velocity field in the region below the accumulation level, and gives a negative contribution to the Reynolds stress, $-\overline{u'w'}$, at levels between z and z_a. The same effect arises from the downward-moving waves from the collapsing patches above the middle of the channel. In the regions $l_m \ll z \ll \frac{1}{2}H$ in which $\bar{u}_z \sim q/\tau$ and $\bar{\rho}_z \sim q^2z^2/\tau^2 l_m^2$, (77) yields

$$z_a = z\left(1 + C\frac{l'}{l_m}\right), \tag{78}$$

where C is of order one. Since $l' \sim l_m$, the waves originating near the plates are absorbed near the plates. However, those originating in the collapsing patches at height z are absorbed at greater heights. If we now assume a tendency for the shear to increase slightly near the middle, perhaps because of a decreasing turbulent transport of momentum, there will be a tendency to collect waves from above and below in this region. All such waves yield positive values of $\overline{u'w'}$ so this will tend to increase the existing negative $\overline{u'w'}$ near the middle. The result is a $(\overline{u'w'})_{zz} < 0$ near the middle. Thus $\bar{u}_{zt} > 0$ and the shear will increase at an accelerating rate.

We would expect that the local $\bar{u}_z^2$ would increase faster than $|\bar{\rho}_z|$, so that the middle layer would ultimately become unstable, break down into Kelvin–Helmholtz instability and thereby yield the required heat transport. To explore this possibility, let us assume that variations of τ in and near the middle layer are of order τ itself. Then

$$\bar{u}_{zt} = \tau_{zz} \sim \frac{\tau}{l_d^2}, \tag{79}$$

where l_d is the thickness of the layer. Also, since $q = 0$ near the middle, we have

$$\bar{\rho}_{zt} = q_{zz} \sim \frac{q}{l_d^2}. \tag{80}$$

The local time-rate-of-change of the Richardson number is

$$\frac{\partial Ri}{\partial t} = -\frac{q_{zz}}{\bar{u}_z^2} - \frac{2|\bar{\rho}_z|\tau_{zz}}{\bar{u}_z^3}, \tag{81}$$

where
$$Ri = \frac{|\bar{\rho}_z|}{\bar{u}_z^2}, \quad \tau = -\overline{u'w'}, \quad q = -\overline{w'\rho'}. \tag{82}$$

In the layer, $q_{zz} < 0$ and $\tau_{zz} > 0$, so that the two terms on the right of (81) are of opposite signs. However, the ratio of the second term to the first term is

$$\frac{\tau\bar{\rho}_z}{q\bar{u}_z}. \tag{83}$$

If $\bar{\rho}_z$ and $\bar{u}_z$ have the same orders of magnitude as in the region $l_m \ll z \ll H$, (83) is of order H^2/l_m^2, so that the second term in (81) dominates and Ri tends to decrease as required for the ultimate instability of the layer. The resulting turbulence will transfer the heat and momentum through the layer despite the inflexion point in the density profile.

In natural circumstances, it appears that layers of strong shear and density gradient will tend to form wherever the curvature of the density profile changes sign. This may explain the general layered nature of the atmosphere and oceans.

While developing the ideas in this paper, I have profited from many discussions with Dr Pierre Welander of the University of Gothenburg. Indeed, Dr Welander was the first to arouse my interest in the subject. Part of the work was supported by the U.S. Department of Commerce, Environmental Science Services Administration under Grant E22-15-69(G) and Grant E22-36-69(G).

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